MEL1002 End-Sem Solution

1. (a) For particle 1: $T_1 = \frac{1}{2} m\dot{x}^2$ 1 Mark

For particle 2:

$$X_2 = x + l\cos\theta \quad \Rightarrow \quad \dot{X}_2 = \dot{x} - l\sin\theta\,\dot{\theta}$$

$$Y_2 = l\sin\theta \quad \Rightarrow \quad \dot{Y}_2 = l\cos\theta\,\dot{\theta}$$ 1 Mark

$$\dot{X}_2^2 + \dot{Y}_2^2 = \dot{x}^2 + l^2\dot{\theta}^2 - 2\dot{x}l\dot{\theta}\sin\theta$$

$$T_2 = \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2 - 2l\sin\theta\,\dot{x}\dot{\theta}\right)$$ 1 Mark

Total Kinetic Energy $= T = T_1 + T_2 = m\dot{x}^2 + \frac{1}{2} m l^2\dot{\theta}^2 - ml\sin\theta\,\dot{x}\dot{\theta}$

(b) Potential Energy of the system: (Datum is particle 1 horizontal position)

$$V = mgl\sin\theta$$

Lagrangian $L = T - V = m\dot{x}^2 + \frac{1}{2} m l^2\dot{\theta}^2 - ml\sin\theta\,\dot{x}\dot{\theta} - mgl\sin\theta$ 1 Mark

(c) $P_x = \frac{\partial L}{\partial \dot{x}} = 2m\dot{x} - ml\sin\theta\,\dot{\theta}$ 1 Mark

$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - ml\sin\theta\,\dot{x}$ 1 Mark

2. The generalized coordinates are

$(R_1, \theta_1, \phi_1, x_2, y_2, z_2, x_3, y_3, z_3)$; No. of GCs = 9

There are 3 Euclidean distance constraints

Let us first write the (transform) coordinates of particle 1 from spherical to Cartesian

$x_1 = R_1\cos\phi_1\cos\theta_1$, $y_1 = R_1\cos\phi_1\sin\theta_1$, $z_1 = R_1\sin\phi_1$, 1 Mark

Constraints are 3 holonomic ones

$$\phi_1: (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 - l_1^2 = 0$$

$$\phi_2: (x_2-x_3)^2 + (y_2-y_3)^2 + (z_2-z_3)^2 - l_2^2 = 0$$

$$\phi_3: (x_3-x_1)^2 + (y_3-y_1)^2 + (z_3-z_1)^2 - l_3^2 = 0$$

2 Mark

D.O.F = No. of GCs – No. of Independent Constraint Eqs.

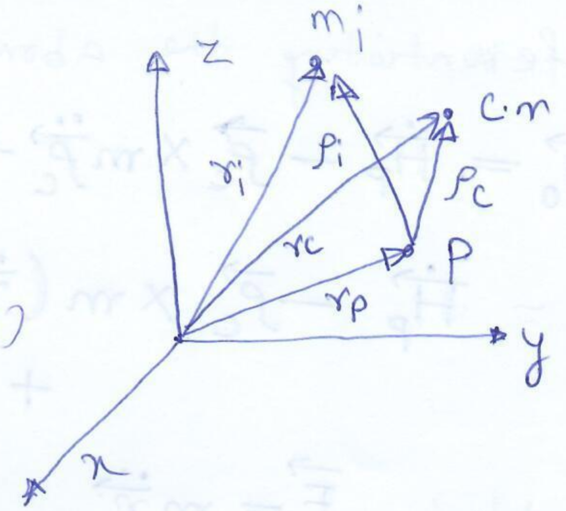
D.O.F. = 9 - 3 = 6

1 Mark

3. We have

$$\vec{r}_i = \vec{r}_p + \vec{\rho}_i$$

$$\vec{r}_c = \vec{r}_p + \vec{\rho}_c$$

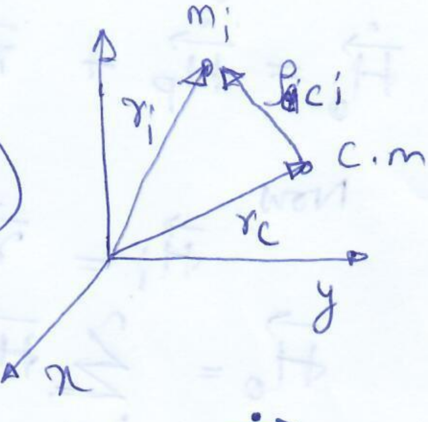


Given

$$\vec{H}_p = \vec{\rho}_c \times m\dot{\vec{\rho}}_c + \vec{H}_c \quad -(i)$$

Angular momentum about the center of mass

$$\vec{r}_i = \vec{r}_c + \vec{\rho}_{ci} \quad \text{— 1 Mark}$$



$$\vec{H}_0 = \sum_{i=1}^{n} (\vec{r}_c + \vec{\rho}_{ci}) \times m_i (\dot{\vec{r}}_c + \dot{\vec{\rho}}_{ci})$$

$$\vec{H}_0 = \vec{r}_c \times m\vec{v}_c + \vec{r}_c \times \sum_{i=1}^{n} m_i \dot{\vec{\rho}}_{ci}$$

$$+ \sum_{i=1}^{n} m_i \vec{\rho}_{cj} \times \dot{\vec{r}}_c + \sum_{i=1}^{n} \vec{\rho}_{cj} \times m_i \dot{\vec{\rho}}_{cj}$$

$$\sum_{i=1}^{n} m_i \vec{\rho}_{ci} = \vec{0} \quad \& \quad \sum_{i=1}^{n} m_i \dot{\vec{\rho}}_{ci} = 0$$

$$\vec{H}_0 = \vec{r}_c \times m\vec{v}_c + \sum_{i=1}^{n} \vec{\rho}_{ci} \times m_i \dot{\vec{\rho}}_{ci}$$

$$= \vec{r}_c \times m\vec{v}_c + \vec{H}_c \quad -(ii) \quad \text{— 2 Mark}$$

Eliminating H_c from eq. (i)

$$\vec{H}_c = \vec{H}_p - \vec{\rho}_c \times m\dot{\vec{\rho}}_c \quad -\text{eq.(iii)} \quad \text{— 1 Mark}$$

Substituting eq.(iii) into eq. (ii)

$$\vec{H}_0 = \vec{r}_c \times m\vec{v}_c + \vec{H}_p - \vec{\rho}_c \times m\dot{\vec{\rho}}_c$$

$$\vec{H}_0 = \vec{H}_p - \vec{\rho}_c \times m\dot{\vec{\rho}}_c + \vec{r}_c \times m\dot{\vec{r}}_c \quad \text{— 1 Mark}$$

Differentiating the above equation

$$\dot{\vec{H}}_0 = \dot{\vec{H}}_P - \vec{\rho}_c \times m\ddot{\vec{\rho}}_c + \vec{r}_c \times m\ddot{\vec{r}}_c$$

$$= \dot{\vec{H}}_P - \vec{\rho}_c \times m(\ddot{\vec{r}}_c - \ddot{\vec{r}}_P) + (\vec{r}_P + \vec{\rho}_c) \times m\ddot{\vec{r}}_c$$ — 1 Mark

Using $\vec{F} = m\ddot{\vec{r}}_c$

$$\dot{\vec{H}}_0 = \dot{\vec{H}}_P + \vec{r}_P \times \vec{F} + \vec{\rho}_c \times m\ddot{\vec{r}}_P$$ — 1 Mark

Now,

$$\vec{H}_i = \vec{r}_i \times m\vec{v}_i$$

$$\vec{H}_0 = \sum_{i=1}^{n} \vec{H}_i = \sum_{i=1}^{n} r_i \times m_i v_i$$

Differentiating with respect to time

$$\dot{H} = \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{v}}_i + \sum_{i=1}^{n} \dot{\vec{r}}_i \times m_i \vec{v}_i$$

Since $\dot{r}_i = v_i$

$v_i \times v_i = 0$

$$\dot{H} = \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{v}}_i \qquad \because m_i \dot{v}_i = F_i + \sum_{j=1}^{n} f_{ij}$$

$$\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_i \times \vec{f}_{ij}$$ — 2 Mark

Internal forces occur in equal and opposite pair

$$\vec{f}_{ij} = -\vec{f}_{ji} \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_i \times \vec{f}_{ij} = 0$$ — 1 Mark

Thus $\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$

Define the total moment about the fixed point O $\quad \vec{M}_0 = \sum \vec{r}_i \times \vec{F}_i \Rightarrow \vec{M}_0 = \dot{\vec{H}}$ — 1 Mark

Hence

$$\vec{M}_0 = \dot{\vec{H}}_P + \vec{r}_P \times \vec{F} + \vec{\rho}_C \times m\ddot{\vec{r}}_P$$

or $$\boxed{\vec{M}_0 = \dot{\vec{H}}_P + \vec{r}_P \times m\ddot{\vec{r}}_C + \vec{\rho}_C \times m\ddot{\vec{r}}_P}$$

proved.

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$$\vec{V}_A = \vec{V}_B + \vec{\Omega} \times \vec{r} + \vec{v}_{rel}$$ 1 Mark

$$= 0 + \omega \hat{k} \times 0.125 \hat{i} + u \hat{i}$$

$$= 0.125 \omega \hat{j} + 0.2 \hat{i}$$

$$= 0.5 \hat{j} + 0.2 \hat{i} \quad m/s$$ 2 Mark

1 Mark

$$\vec{a}_A = \vec{a}_B + \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2 \vec{\Omega} \times \vec{v}_{rel}$$

$$= 0 + 0 + \omega \hat{k} \times (\omega \hat{k} \times 0.125 \hat{i}) + 2 (\omega \hat{k} \times 0.2 \hat{i})$$

$$= -0.125 \omega^2 \hat{i} + 2 (\omega \hat{k} \times 0.2 \hat{i})$$

$$= -2 \hat{i} + 2 (4 \hat{k} \times 0.2 \hat{i})$$

$$= -2 \hat{i} + 1.6 \hat{j} \quad m/s^2$$ 2 Mark

Absolute acceleration

$$|\vec{a}| = a_A = \sqrt{2^2 + 1.6^2} = 2.56 \ m/s^2$$

Angular momentum about point B

$$\vec{H}_B = \vec{r} \times m \vec{v}$$ 1 Mark

$$\vec{r} = 0.125 \hat{i}$$

$$m\vec{v} = 0.1 (0.2 \hat{i} + 0.5 \hat{j})$$

$$\vec{H}_B = 0.125 \hat{i} \times 0.05 \hat{j}$$

$$= 0.00625 \hat{k} \ Kg \frac{m^2}{s}$$ 3 Mark

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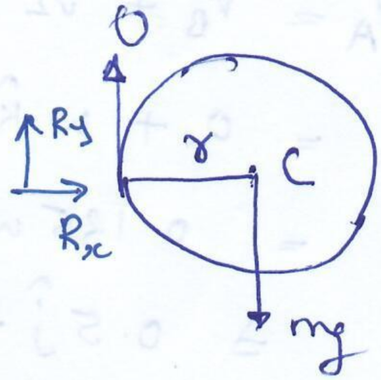
$\sum M_o = I_o \alpha$ — 1 Mark

1 Mark

$mgr = 2mr^2 \alpha$

$\alpha = g/2r$ 2 Mark



$\sum F_y = m\bar{a}_y$ 1 Mark

$\bar{a}_y = \frac{g}{2r} r$ 1 Mark

$\bar{a}_x = 0$

$\Rightarrow -mg + R_y = -\frac{mrg}{2r}$

$R_y = \frac{mg}{2}$ 1 Mark

$\sum F_x = m\bar{a}_x = 0$ 1 Mark $(\because \bar{a}_x = 0)$. 1 Mark

$\Rightarrow R_x = 0$ — 1 Mark

<u>6</u>

$$\omega = \Omega \hat{k} + p\cos\beta\,\hat{k} + p\sin\beta\,\hat{j} - \dot{\beta}\,\hat{i}$$ 2 Mark

$$= -\dot{\beta}\,\hat{i} + (p\sin\beta)\,\hat{j} + (p\cos\beta + \Omega)\,\hat{k}$$ 1 Mark

Given, the entire unit rotates about the vertical Z-axis at a constant angular velocity Ω rad/s

$$\frac{d\omega}{dt} = \underbrace{\frac{d}{dt}\left(-\dot{\beta}\,\hat{i}\right)}_{\text{1st term (I)}} + \underbrace{\frac{d}{dt}\left(p\sin\beta\,\hat{j}\right)}_{\text{2nd term (II)}} + \underbrace{\frac{d}{dt}\left((p\cos\beta + \Omega)\,\hat{k}\right)}_{\text{3rd term III}}$$

(I) $$\frac{d}{dt}\left(-\dot{\beta}\,\hat{i}\right) = -\ddot{\beta}\,\hat{i} - \dot{\beta}\,\dot{\hat{i}}$$

$$= -\dot{\beta}\left(\hat{\Omega} \times \hat{i}\right) \quad \because \dot{\hat{i}} = \Omega \times i$$

$$= -\dot{\beta}\left(\Omega\,\hat{k} \times \hat{i}\right)$$

$$= -\dot{\beta}\,\Omega\,\hat{j}$$ 2 Mark

since p is const

(II) $$\frac{d}{dt}(p\sin\beta)\,\hat{j} = p\,\frac{d}{dt}\sin\beta\,\hat{j} + \cancelto{0}{\dot{p}}\sin\beta\,\dot{\hat{j}}$$

$$= p\left(\cos\beta\,\dot{\beta}\,\hat{j} + \sin\beta\,\dot{\hat{j}}\right)$$

$$= p\cos\beta\,\dot{\beta}\,\hat{j} + \sin\beta\left(\hat{\Omega} \times \hat{j}\right)$$

$$= p\cos\beta\,\dot{\beta}\,\hat{j} + \sin\beta\left(\Omega\,\hat{k} \times \hat{j}\right)$$

$$= p\cos\beta\,\dot{\beta}\,\hat{j} - p\sin\beta\,\Omega\,\hat{i}$$ 2 Mark

(III) $$\frac{d}{dt}\left(p\cos\beta + \Omega\right)\hat{k} = \left(-p\sin\beta\,\dot{\beta} + \cancelto{0}{\dot{p}}\cos\beta + \cancelto{0}{\frac{d\Omega}{dt}}\right)\hat{k}$$

$$= -p\sin\beta\,\dot{\beta}\,\hat{k}$$ 2 Mark

$\boxed{\text{Correct } \vec{\alpha} - 1\text{ Mark}}$

$$\hat{\alpha} = -\Omega p \sin\beta \hat{i} + \dot{\beta}(p\cos\beta - \Omega)\hat{j}$$
$$- p\dot{\beta}\sin\beta\hat{K}$$ 1 Mark

Alternative solution for angular acceleration

$$\left(\frac{d\hat{\omega}}{dt}\right)_{XYZ} = \left[\frac{d\hat{\omega}}{dt}\right]_{xyz} + \hat{\Omega} \times \hat{\omega}$$

$$\because \omega = -\dot{\beta}\hat{i} + p\sin\beta\hat{j} + (p\cos\beta + \Omega)\hat{k}$$

$$\left.\frac{d\hat{\omega}}{dt}\right|_{xyz} = 0 + p\dot{\beta}\cos\beta\hat{j} + (-p\dot{\beta}\sin\beta + 0)\hat{k}$$

$$\hat{\Omega} \times \hat{\omega} = \Omega\hat{k} \times (-\dot{\beta}\hat{i} + p\sin\beta\hat{j} + [p\cos\beta + \Omega)K)$$
$$= -\Omega\dot{\beta}\hat{j} - \Omega p \sin\beta\hat{i}$$

So,
$$\alpha = (p\dot{\beta}\cos\beta - \Omega\dot{\beta})\hat{j} - \Omega p\sin\beta\hat{i} - p\dot{\beta}\sin\beta\hat{k}$$
$$\alpha = -\Omega p\sin\beta\hat{i} + \dot{\beta}(p\cos\beta - \Omega)\hat{j} - p\dot{\beta}\sin\beta\hat{k}$$

7 Mark

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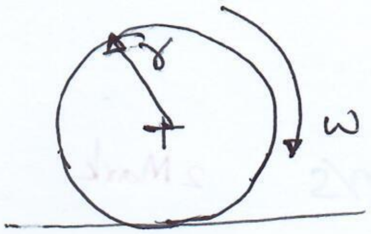

Conserving angular momentum about the point of contact (since friction exerts no torque about that point

1 Mark

Moment of inertia of the disc

$$I_{c.m} = \frac{1}{2} m r^2$$

Applying angular momentum conservation about contact point

Initially pure rotation, no translation

$$L_i = I_{c.m}\,\omega_0 + m v_{c.m} r$$

$$= \frac{1}{2} m r^2 \omega_0 + 0 \quad \because v_{c.m} = 0$$ 2 Mark

Finally it rolls without slipping $v_{c.m} = \omega_f r$

$$L_f = I_{c.m}\,\omega_f + m v_{c.m} r$$

$$= I_{c.m}\frac{v_{c.m}}{r} + m v_{c.m} r$$ 2 Mark

$$\because L_f = L_I$$

$$\Rightarrow \frac{1}{2} m r^2 \omega_0 = v_{c.m}\left(\frac{1}{2} m r + m r\right) \quad \because I_{c.m} = \frac{1}{2} m r^2$$

$$= v_{c.m}\,\frac{3}{2} m r$$

$$v_{c.m} = \frac{1}{2}\,\frac{r^2 \omega_0}{\frac{3}{2} r} = \frac{1}{3} r \omega_0$$

$$\Rightarrow V_{c.m} = \frac{1}{3} \times 0.15 \times 5$$

$$= \frac{0.75}{3} = 0.25\, m/s$$ 2 Mark

Given

$m = 1\,kg$; $r = 0.15\,m$; $v = 0.25\,m/s$

Rolling condition $\omega = v/r = \frac{0.25}{0.15} = 1.67\ rad/s$

Total kinetic energy

$$= \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

Translational

$$K.E = \frac{1}{2} m v^2 = \frac{1}{2} \times 1 \times (0.25)^2$$

$$= 0.03125\ J$$ 1 Mark

Rotational $K.E = \frac{1}{2} I \omega^2$

$$I = \frac{1}{2} \times 1 \times (0.15)^2 = 0.01125$$

$$\therefore \frac{1}{2} I \omega^2 = \frac{1}{2} \times 0.01125 \times (1.67)^2$$ 1 Mark

$$= 0.015625\ J$$

Total $K.E = 0.03125 + 0.015625$

$$= 0.0469\ J$$ 1 Mark